A Variational Principle for Radial Flows in Holographic Theories.

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Abstract

We develop furthur the correspondence between a $d+1$ dimensional theory and a d dimensional one with the "radial" $(d+1)$th coordinate ρ playing the role of an evolution parameter. We discuss the evolution of an effective action defined on a d dimensional surface charactarized by ρ by means of a new variational principle. The conditions under which the flow equations are valid are discussed in detail as is the choice of boundary conditions. It is explained how domain walls may be incorporated in the framework and some generalized junction relations are obtained. The general principles are illustrated on the example of a supergravity theory on AdS_{d+1}.

1 Introduction

The holographic principle [1, 2, 3, 4], motivated by Black Hole considerations, is an extension of the Bekenstein bound [5] which limits the number of degrees of freedom that generate entropy in a theory including gravitation. It is a dynamical statement postulating that the evolution of gravitational and matter fields in bulk space-time is specified by the data stored in its boundary. In the conventional quantum field theoretic treatment of gravity, this property is not apparent, however, recent conjectures in string theory concerning the AdS/CFT duality [6, 7, 8, 9] provide an example of this principle. If future investigations validate it, the holographic principle could well turn out to be one of the most important physical ideas in recent times.

There are many forms that a correspondence between a $d + 1$ dimensional and a d dimensional theory can take. Perhaps the simplest one is that between a classical theory in the higher dimensional space and a quantum theory at the boundary. An example is provided by the conjectured duality between the classical supergravity theory in 5 dimensions and a quantum gauge theory in 4 dimensions [6]. More specifically, let $S(\phi, g)$ denote the classical action for a supergravity solution where the boundary values of the fields are ϕ and of the metric, $g_{\mu\nu}$, then the presciption [7, 8] for the conjectured duality is:

$$\frac{1}{\sqrt{g}} \frac{\delta}{\delta \phi^{i_1}} ...... \frac{1}{\sqrt{g}} \frac{\delta}{\delta \phi^{i_n}} S \ = \ < O_{i_1} ...... O_{i_n} > \tag{1}$$

where, O denote certain gauge invariant operators in the boundary theory. In order for the classical description in the bulk of the AdS_5 space to be valid one considers the limit of large N and large 't Hooft coupling on the gauge theory side. A key ingredient of this correspondence is the UV/IR connection [10, 11] : the infrared regulator of the bulk theory is equivalent to the ultraviolet regulator of the boundary theory. This indicates that one may interpret the $d+1$th coordinate, ρ as a renormalization group parameter of the 4 dimensional theory. Indeed, the evolution of bulk fields with ρ has been studied in a fundamental paper [12] within the Hamilton-Jacobi framework where it was shown that the resulting flow

equations can asymptotically be cast in the form of a Callan-Symanzik equation, with the bulk scalar field representing the 4 dimensional gauge coupling. In ref. [10] the boundary AdS/CFT conjectured correspondence has been extended to a finite radius AdS foliation. This is of relevance to the connection with the Randall-Sundrum type scenarios [13]. A more generalized correspondence was originally proposed [6], i.e., between the type II B string theory in the bulk (of AdS_5 times S_5) and a CFT on the boundary. In the limit of large N and large 't Hooft coupling, the string coupling $g \to 0$ and one may limit to the classical supergravity description, otherwise string loops need to be considered.

Though much work has gone into the development of these ideas, there are many challenges ahead [9]. Of particular relevance for this paper is the work of [12] concerning the evolution of bulk fields with ρ. We will formulate this evolution quite generally by means of a variational principle. The advantages of doing so are many : a variational principle replaces many mathematical expressions by a single general principle and moreover the formulation is coordinate independant. The variational principle which we derive in section 2 involves a multi-stage optimization procedure -a minimization condition which leads to a flow equation for the effective action S. The effective action is a function of the fields on a d dimensional surface charactarized by a particular value of ρ. As one changes ρ the action changes and if we still require a minimum of the action integral, then the flow equations must be satisfied. This procedure is discussed in detail in section 2 where its connection with the Hamilton-Jacobi formalism is also shown. The problem is first discussed in a general setting and then the generalized flow equations are applied to the example of the supergravity theory on AdS_{d+1} where the flow equations of ref.[12] are reproduced. In addition to the above, a variational formulation provides us with a new way to formulate the boundary conditions which the flow equations must satisfy. This is done in section 3 together with a discussion of the Weyl anomaly. Another decided advantage of the variational formulation is discussed in section 4. There we discuss in a quantitative manner, the conditions under which the "classical" approximation for the flow equations is valid. In particular, we seek the conditions under

which terms involving the second partial derivatives of the effective action which are ignored in the derivation of the flow equations in section 2 can become unbounded. This allows us to formulate the applicability criterion quite generally in terms of (the solutions of) a set of partial differential equations. The example of a single scalar field in AdS is then considered where the differential equation reduces to a Jacobi type equation. The question of how the second partial derivatives of the action may be incorporated into the flow equations in this framework will be dealt with in a subsequent publication [14]. Our formulation, as with ref.[12], allows one to consider the effective action for values of ρ away from the boundary. In the AdS case, for example, we can consider any of its foliations. This allows us to study the scenario when a domain wall is introduced into the $d + 1$ dimensional space-time. This is done quite generally in section 5 where starting from the results of section 2, generalized junction conditions [15] are derived. The example of the Randall-Sundrum [13] scenario and its connection with the supergravity theory on AdS [16] is also discussed in some detail. Finally, in section 6 we conclude with a discussion of the results.

2   The Variational Principle

In this section we will consider the dynamical evolution of fields in a $d + 1$ dimensional spacetime with a d dimensional (time-like) boundary. We will denote the d dimensional coordinates by x_μ and the "radial" coordinate is ρ. We consider a d dimensional hypersurface defined by $\rho = $ constant after making appropriate gauge choices for the metric. In the AdS case, for example, this hypersurface would be a particular foliation. The boundary of the $d+1$ dimensional space is located at $\rho = \rho_0$. We will be interested in fields $\phi_i(x, \rho)$ [17] whose dynamics is governed by a lagrangian $\mathcal{L}$. In particular, we would like to understand how the effective action changes as we move from one hypersurface to a neighbouring one. The trajectory that one follows in moving from one hypersurface to the other is not arbitrary but is determined by an optimization condition. We propose below that this optimal trajectory

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is determined by a variational principle which will be seen to imply a (functional) partial differential equation satisfied by the effective action $S(\phi_i)$ of the boundary values of the fields.

The optimal trajectory arises as a result of a minimization problem and the solution suggests itself as a consequence of a multi-stage procedure. To see how this works, consider first the integral:

$$J'[\phi_i] = \int_{\rho_1}^{\rho_2} d\rho F(\rho, \phi_i, \dot{\phi}_i) \qquad (2)$$

In the above, $\dot{\phi}_i = d\phi_i/d\rho$. In general, $F(\rho, \phi_i, \dot{\phi}_i)$ will be of the form, $\int_d \sqrt{G}\mathcal{L}(\rho, \phi_i, \dot{\phi}_i)$, with $\mathcal{L}$ the Lagrangian, and $G_{\mu\nu}$ is the $d+1$ dimensional metric. In this and the following we suppress any dependance on the d dimensional coordinates. We will call the critical trajectory, the one which is obtained by minimizing J'. Consider a trajectory, and let us choose a point (ρ_3, ϕ_{i3}) [18] in between the initial and final points, and follow the curve from (ρ_1, ϕ_{i1}) to (ρ_3, ϕ_{i3}). For the rest of the curve $((\rho_3, \phi_{i3})$ to (ρ_2, ϕ_{i2}) to be also critical we must minimize $\int_{\rho_3}^{\rho_2} d\rho F(\rho, \phi_i, \dot{\phi}_i)$. This must be true for all such inbetween points ρ_3. Thus whatever the initial point (ρ_n, ϕ_{in}), or the initial arc of the trajectory, the remaining transformations must constitute an optimal sequence for the remaining problem. The endpoint of the first transformation is the initial point of the next and so on. The key point here is that the perturbation in the first interval produces a dependant deformation of the remaining curve. Let us next apply such a multi-stage procedure to the variational problem of minimizing J'.

In view of the above comments, let us consider instead of Eq. (1), the following integral

$$J[\phi_i] = \int_{\rho_1}^{\rho} d\rho' F(\rho', \phi_i(\rho'), \dot{\phi}_i(\rho')). \qquad (3)$$

Suppose that for a given (ρ, ϕ_i) there is some trajectory that minimizes J. Let us denote

$$S(\rho, \phi_i) = Min\left\{\int_{\rho_1}^{\rho} d\rho' F(\rho', \phi_i(\rho'), \dot{\phi}_i(\rho'))\right\} \qquad (4)$$

In order to use the idea presented above, we first divide the interval thus:

$$(\rho_1, \rho) = (\rho_1, \rho - \Delta\rho) + (\rho - \Delta\rho, \rho). \qquad (5)$$

Along the arc associated with the first interval we take the trajectory to be critical and for the second interval it is, in general, arbitrary except that at the endpoint $\phi_i(\rho) = \phi_i$. The corresponding contributions to the integral J are denoted by J_1 and J_2;

$$J_1 = \int_{\rho_1}^{\rho - \Delta\rho} d\rho' F(\rho', \phi_i(\rho'), \dot{\phi}_i(\rho')) \tag{6}$$

and,

$$J_2 = \int_{\rho - \Delta\rho}^{\rho} d\rho' F(\rho', \phi_i(\rho'), \dot{\phi}_i(\rho')). \tag{7}$$

Since the trajectory is critical in the first interval, we get using the definition in Eq.(4),

$$J_1 = S(\rho - \Delta\rho, \phi_i(\rho - \Delta\rho)) \tag{8}$$

Expanding to first order in $\Delta\rho$,

$$J_1 = S(\rho - \Delta\rho, \phi_i - \dot{\phi}_i \Delta\rho) \tag{9}$$

An important point to notice here is that the function $\dot{\phi}_i$ is arbitrary above and in the following expression for J_2. Since the second interval is infinitesmal, we have again to first order in $\Delta\rho$

$$J_2 = F(\rho, \phi_i, \dot{\phi}_i)\Delta\rho. \tag{10}$$

Since the sum of J_1 and J_2 is greater than or equal to S, and since in Eq.(9) and Eq.(10) $\dot{\phi}_i$ is the arbitrary function, we have:

$$S(\rho, \phi_i) = \overset{Min}{\dot{\phi}_i} \left\{ F(\rho, \phi_i, \dot{\phi}_i)\Delta\rho + S(\rho - \Delta\rho, \phi_i - \dot{\phi}_i \Delta\rho) \right\} \tag{11}$$

Assuming that the second and higher partial derivatives of S are bounded we can again neglect higher orders in $\Delta\rho$ to get,

$$S(\rho, \phi_i) = \overset{Min}{\dot{\phi}_i} \left\{ F(\rho, \phi_i, \dot{\phi}_i)\Delta\rho + S(\rho, \phi_i) - \frac{\partial S}{\partial \rho}\Delta\rho - \frac{\delta S}{\delta \phi_i}\dot{\phi}_i \Delta\rho \right\}. \tag{12}$$

Finally, in the limit of $\Delta\rho \to 0$ we obtain:

$$\underset{\dot{\phi}_i}{Min} \left\{ F(\rho, \phi_i, \dot{\phi}_i) - \frac{\partial S}{\partial \rho} - \frac{\delta S}{\delta \phi_i} \dot{\phi}_i \right\} = 0. \tag{13}$$

This is the main result of this section. It is worth emphasizing here that, $\dot{\phi}_i(\rho, \phi_i)$, the value of which at each point (ρ, ϕ_i) minimizes the expression on the right hand side of the above equation is associated with a solution $S(\rho, \phi_i)$ of that equation. If S is known then its partial derivatives are easily obtained and the value of the "flow velocities" $\dot{\phi}_i(\rho, \phi_i)$ at any point can be determined by minimizing a function of only the variables $\dot{\phi}_i$; conversely, if $\dot{\phi}_i(\rho, \phi_i)$ are known everywhere then S can be obtained by evaluating the integral J in Eq.(3) with the given flows. Thus knowledge of the flow velocities or the action along the optimal trajectory constitutes a complete solution to the problem.

From the above, it follows that at the minimum,

$$F(\rho, \phi_i, \dot{\phi}_i) = \frac{\partial S}{\partial \rho} + \frac{\delta S}{\delta \phi_i} \dot{\phi}_i \tag{14}$$

and in addition,

$$\frac{\delta F}{\delta \dot{\phi}_i} = \frac{\delta S}{\delta \phi_i} \tag{15}$$

It should be noted that Eq.(15) asserts that for a given numerical value of $\rho, \phi_i, \frac{\partial S}{\partial \rho}, \frac{\delta S}{\delta \phi_i}$, the derivative w.r.t. $\dot{\phi}_i$ of Eq.(13) must be zero when $\dot{\phi}_i$ minimizes the bracketed quantity. The above two equations imply the Hamilton Jacobi equation ($H + \frac{\partial S}{\partial \rho} = 0$), for this system as can be easily seen by noting that the Hamiltonian is given by:

$$H = -F + \dot{\phi}_i \frac{\delta F}{\delta \dot{\phi}_i}. \tag{16}$$

In theories including gravity, since the local shifts $\rho \to \rho + \Delta\rho$ are part of the general coordinate invariance, we have the implied constraint , $H = 0$. In this case it is the following (functional) differential equation which determines the action functional S:

$$F(\rho, \phi_i, \dot{\phi}_i) = \frac{\delta S}{\delta \phi_i} \dot{\phi}_i \tag{17}$$

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and in addition the momenta conjugate to the fields are given by:

$$\Pi_i = \frac{1}{\sqrt{g}} \frac{\delta F}{\delta \dot{\phi}_i} \tag{18}$$

Eqs.(15,18) allow us to relate the momenta with the flow velocities $\dot{\phi}_i$ of the fields. Since the effective action, $S = S(\phi_i, \rho)$ only, we see that these equations also imply that we may in principle solve for the $\dot{\phi}_i$ in the form:

$$\dot{\phi}_i = g_i(\phi_i, \frac{\delta S}{\delta \phi_i}, \rho) = f_i(\phi_i, \rho). \tag{19}$$

Next we briefly discuss the relation of the fundamental Eq.(13) or Eqs.(14, 15) to the $d+1$-dimensional equations of motion. Henceforth we will adopt the notation (the derivatives may be functional or ordinary partials),

$$S_{,y_1 y_2} \equiv \frac{\delta^2 S}{\delta y_1 \delta y_2}, \tag{20}$$

and so on. Along the critical trajectory, consider

$$(dS_{,\phi_i}/d\rho)_{\dot{\phi}_k} = S_{,\rho\phi_i} + S_{,\phi_i\phi_j}\dot{\phi}_j \tag{21}$$

Similarly, differentiating Eq.(14) with respect to ϕ_i gives,

$$F_{,\phi_i} + (F_{,\dot{\phi}_j} - S_{,\phi_j})\frac{\delta\dot{\phi}_j}{\delta\phi_i} - S_{,\rho\phi_i} - S_{,\phi_i\phi_j}\dot{\phi}_j = 0. \tag{22}$$

We will discuss the possibility of $\dot{\phi}_i$ being discontinuous along a critical trajectory in a later section, hence omitting this possibility for the moment and using Eq.(15) and Eq.(21) we get,

$$(dS_{,\phi_i}/d\rho)_{\dot{\phi}_k} = F_{,\phi_i}. \tag{23}$$

In a similar manner and under similar assumptions we can also get:

$$(dS_{,\rho}/d\rho)_{\dot{\phi}_k} = F_{,\rho} \tag{24}$$

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Using Eq.(15), it is seen that Eq.(23) is in fact the $d+1$ dimensional Euler Lagrange equation:

$$(dF_{,\dot\phi_i}/d\rho)_{\dot\phi_k} \;=\; F_{,\phi_i}, \tag{25}$$

which are seen to be satisfied along the critical trajectory.

We have seen that corresponding to the critical trajectory there exist partial derivatives of the action S which satisfy the differential equations (23,24) such that the flow velocity along the critical trajectory minimizes for each value of ρ the expression :

$$F \;-\; S_{,\rho} \;-\; S_{,\phi_i}\dot\phi_i, \tag{26}$$

and in addition that minimum value is zero. This allows for an interpretation which is in the spirit of the renormalization group approach. The "cut-off" corresponds to a particular value of ρ. As we change the cut-off, we continue to require that the trajectory is critical and this gives us Eq.(13). The variational principle thus can be thought of as an alternative formulation of the renormalization group flow. However up until now we are restricting ourselves to the classical theory in the bulk. As discussed in Sec.[4] the above treatment breaks down when the second derivatives of S cannot be neglected in the derivation leading upto Eq.(13). Then a more general (quantum) treatment is required, which is developed in ref.[14]. As mentioned earlier, in $d+1$ dimensional theories with gravity we must set the gauge constraint: $S_{,\rho} \;=\; 0$. In such theories, of interest to the holographic principle, the relevant equations are (17), (15) and (18).

To summarize let us express the results of this section in a geometrical setting. At all points other than the initial point the hypersurface (foliation) in the $d+1$ dimensional space may be denoted by:

$$S(\phi_i, \rho) \;=\; \alpha, \tag{27}$$

where α is a parameter. We have also seen that along the critical trajectory, Eq.(15), is satisfied. As discussed earlier, it allows us to solve for $\dot\phi_i$ through Eq.(19). The curves so defined intersect the hypersurface, the former being determined by the latter. Let $P(\phi_i, \rho)$

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denote a point on the hypersurface $S = \alpha$. A critical curve passes through P and will intersect the neighbouring hypersurface $S = \alpha + d\alpha$ at some point $Q(\phi_i + \delta\phi_i, \rho + \delta\rho)$, where $\delta\phi_i = \dot\phi_i d\rho = f_i(\phi_i, \rho)d\rho$. The displacement PQ is, $d\alpha = Fd\rho$, which is independant of the position of P on the first hypersurface $S = \alpha$. Thus along the critical trajectory when we go from one hypersurface to another, the increment of the fundamental integral is always $d\alpha$. Generalizing this we may integrate from $S = \alpha_1$ to $S = \alpha_2$ along the critical trajectory :

$$\int_{P_1}^{P_2} Fd\rho = \int_{P_1}^{P_2} \left(\frac{\partial S}{\partial \rho} + \frac{\delta S}{\delta\phi_i}\dot\phi_i\right) = \alpha_2 - \alpha_1, \tag{28}$$

which is independant of the position of P_1. Thus we have an interesting result that a family of critical curves will cut off "equal distances" between two such surfaces.

In order to illustrate the above ideas let us consider in some detail a specific example in the framework of the AdS_{d+1}/CFT_d correspondence. As usual, we will consider a supergravity theory on AdS_{d+1} which contains scalars ϕ^I. This supergravity theory is the low energy limit of the type IIB string theory and the boundary CFT it is conjecturaly equivalent to is a $\mathcal{N} = 4$ SYM theory in the limit of large N and large 't Hooft coupling. The action for the bulk theory is (setting the $d + 1$ dimensional Newton's constant to unity) :

$$I_{d+1} = \int_{d+1} (R^{d+1} + 2\Lambda) + 2\int_d K + \int_{d+1} (V(\phi) + \frac{1}{2}G_{IJ}\nabla_a\phi^I\nabla^a\phi^J), \tag{29}$$

where, the $d + 1$ dimensional cosmological constant $2\Lambda \equiv \frac{d(d-1)}{r^2}$, and r is the AdS radius, V and G_{IJ} are the $d + 1$ dimensional scalar potential and metric and K is the trace of the extrinsic curvature, $K_{\mu\nu}$ of an arbitrarily chosen foliation at $\rho = $ constant. For reasons of uniqueness of solutions [8] we henceforth take the $d+1$ dimensional space to have Euclidean signature with a metric :

$$ds^2 = d\rho^2 + g_{\mu\nu}(x, \rho)dx^\mu dx^\nu, \tag{30}$$

which entails a specific gauge choice. For large ρ the boundary metric $g_{\mu\nu} \sim e^{2\rho/r}$ so infinities arise as we take the boundary to infinity. these must be cancelled with counterterms [8, 19, 20, 21] before the limit is taken. It has been dicussed previously that crucial to the duality

of the bulk and boundary thoeries is the UV/IR connection. With the choice of gauge (30), the extrinsic curvature of a foliation at ρ is,

$$K_{\mu\nu} = \frac{1}{2}\dot{g}_{\mu\nu}. \tag{31}$$

Using the Gauss-Cordacci equations one obtains :

$$I_{d+1} = \int d^d x d\rho \sqrt{g}(R^d + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\Lambda + V(\phi) + \frac{1}{2}G_{IJ}\nabla_a \phi^I \nabla^a \phi^J). \tag{32}$$

With the identification,

$$F = \int d^d x (\sqrt{g}\mathcal{L}) \tag{33}$$

$$\sqrt{g}\mathcal{L} = \sqrt{g}(R^d + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\Lambda + V(\phi) + \frac{1}{2}G_{IJ}\nabla_\mu \phi^I \nabla^\mu \phi^J); \tag{34}$$

the canonical momenta conjugate to $g^{\mu\nu}$ and ϕ^I are,

$$\pi_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta F}{\delta \dot{g}^{\mu\nu}} = K_{\mu\nu} - Kg_{\mu\nu} \tag{35}$$

$$\pi_I = \frac{1}{\sqrt{g}}\frac{\delta F}{\delta \dot{\phi}^I} = G_{IJ}\dot{\phi}^J. \tag{36}$$

The above and equations (15) now give the flow velocities as :

$$\dot{g}_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} - \frac{2}{d-1}g_{\mu\nu}g^{\alpha\beta}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}} \tag{37}$$

$$\dot{\phi}^I = G^{IJ}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta \phi^J}. \tag{38}$$

The master equation (17) for this case can hence can be written as,

$$\frac{1}{\sqrt{g}}\left(-\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta S}{\delta g^{\mu\nu}} - \frac{1}{2}\frac{\delta S}{\delta \phi^I}\frac{\delta S}{\delta \phi^J}G^{IJ} + \frac{1}{d-1}\left(g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}}\right)^2\right) = \sqrt{g}(R^d + 2\Lambda + V(\phi) + \frac{1}{2}G_{IJ}\nabla_\mu \phi^I \nabla^\mu \phi^J) \tag{39}$$

This equation, first obtained in [12], determines the form of the effective action as a function of the boundary fields ϕ_I and $g_{\mu\nu}$ when the boundary is at ρ. Changing ρ changes the boundary and the effective action also changes in such a manner that one is still on a critical

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trajectory. As emphasized before, the $d + 1$ coordinate ρ plays the role of a renormalization group parameter and it was shown [12] that in the asymptotic limit the above equation can be cast in the form of a Callan-Symanzik equation. This provides justification for the identification of S with the quantum effective action of a d dimensional theory at the boundary of AdS_{d+1}.

Equation(39) can be used to determine the form of the effective action. In general, at some energy scale, S may be split into a local and a non-local part and a derivative expansion of the former performed . The different terms in the derivative expansion are constrained by (39), and in fact thier coefficients can be thus determined. Including terms upto four derivatives we can write an ansatz for S_{loc} as :

$$S_{loc} = \int_d \sqrt{g} \left(\Phi_1(\phi) R + U(\phi) + \Phi_2(\phi) R^2 + \Phi_3(\phi) R_{\mu\nu} R^{\mu\nu} + \Phi_4(\phi) R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \right) \qquad (40)$$

Then we get for the coefficients of the leading two terms in (40),

$$U^2 - \frac{4(d-1)}{d} \frac{1}{2} G^{IJ} U_{,IJ} = \frac{4(d-1)}{d} 2\Lambda + \frac{4(d-1)}{d} V \qquad (41)$$

$$\frac{d-2}{2(d-1)} U \Phi_1 - G^{IJ} U_{,IJ} = 1 \qquad (42)$$

In addition, for the case of pure gravity ($V = 0$, $\delta_\mu \phi_I = 0$) we get the following for the coefficients of the higher derivative terms [22],

$$\Phi_2 = -\frac{1}{2} \frac{d}{d-4} \frac{\Phi_1^2}{U}, \qquad (43)$$

$$\Phi_3 = 2 \frac{d-1}{d-4} \frac{\Phi_1^2}{U}, \qquad (44)$$

$$\Phi_4 = 0. \qquad (45)$$

We will make use of these results in a future section.

3 Boundary Conditions.

In the context of holographic theories, specially the AdS/CFT example, several types of boundary conditions have been proposed for the fields on the boundary. In this section we

will consider this question from the perspective of the differential equations (14, 17) and (15), for the effective action S. The proper choice of the boundary conditions on $S(\rho, \phi_i)$, in contrast to Eqs.(14, 17), does depend upon the initial conditions on the allowed trajectories. We will consider some of the possibilities below.

If the allowed trajectories are those that start at any point on the "surface" (read foliation for the AdS example) $\rho = \rho_1$, then if the function $F(\rho, \phi_i, \dot{\phi}_i)$ does not develop a divergence at that point, we have $S(\rho_1, \phi_i) = 0$ for all ϕ_i. This follows from Eq.(4) where at $\rho = \rho_1$, the upper and lower limits coincide. Note that for this choice of boundary conditions $\phi_i(\rho_1)$ is unspecified. It is also important to realize that with this choice of boundary conditions, $S_{,\phi_i}$ and $S_{,\phi_i\phi_j}$ are also identically zero along the "surface" $\rho = \rho_1$. We should mention a straightforward generalization of this case, i.e., if the initial condition is such that $\phi_i(\rho_1) = f_i(\rho_1)$, then $S(\rho, \phi_i) = 0$ for all ρ, ϕ_i such that $\phi_i = f_i(\rho)$, for the same reasons as before.

We next discuss a different choice of boundary conditions which is more relevant to applications of the holographic principle. This is the situation that all allowed critical curves begin at a specified initial point $(\rho_1, \phi_i(\rho_1))$. Since $\phi_i(\rho_1)$ is specified, no other points at $\rho = \rho_1$ with ϕ_i having any other value is allowed. Then, as S is defined along the "surface" $\rho = \rho_1$ only at $\phi_i = \phi_{i1}$ at which point $S(\rho_1, \phi_{i1}) = 0$, we see that the effective action does not have a well defined partial derivative with respect to ϕ_i at this point. Along the critical curve, in fact, $S_{,\phi_i}$ is well defined at all (ρ, ϕ_i) except at the initial point. As we approach an arbitrarily close final point from the initial point, along the critical curve, $S_{,\phi_i}$ is well defined and as will be shortly seen, this allows us to obtain a limiting value for this quantity at (ρ_1, ϕ_{i1}). Of course, this limiting value of $S_{,\phi_i}$ will be different for different critical curves emanating from the same initial point but ending at different final points.

Indeed, we have seen in the previous section that along the critical curve, Eq.(15) is satisfied. This is true at all points except at the initial point where $S_{,\phi_i}$ is not well defined. However, we can extrapolate back from an arbitrarily close point along the curve to the initial point assuming a "straight -line" path. Thus from a point (ρ, ϕ_i) very close to the

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initial point, (ρ_1, ϕ_{i1}) we approximate the critical curve by a "straight line" and write:

$$\dot{\phi}_i = \frac{\phi_i - \phi_{i1}}{\rho - \rho_1}. \tag{46}$$

Now we can evaluate $F_{,\dot{\phi}_i}$ using the above, and demand that Eq.(15) is satisfied as the reference point (ρ, ϕ_i) approaches the initial point. The picture of the allowed trajectories that thus emerges is the following: The critical curves all start from the same initial point (ρ_1, ϕ_{i1}) and fan out each with different values of the initial "slope" $S_{,\phi_i}$ arriving at a boundary manifold at different points. As seen from the boundary the various critical trajectories come from a "focal point" (the initial point) in the bulk of the $d+1$ dimensional space-time.

The limiting behaviour of $\frac{\delta \dot{\phi}_i}{\delta \phi_j}$ as (ρ, ϕ_i) approaches the initial point can also be obtained for this choice of boundary conditions from the following argument : Expanding ϕ_{i1} in a Taylor series, about $\phi_i(\rho)$ which is close to it, we have,

$$\phi_i(\rho_1) = \phi_i(\rho) + (\rho_1 - \rho)\dot{\phi}_i(\rho) + ... \tag{47}$$

We now replace functional derivatives with ordinary derivatives by using the definition,

$$\int \frac{\delta}{\delta \phi_i} = \frac{\partial}{\partial \phi_i}. \tag{48}$$

Since ϕ_{i1} is fixed, we get upon differentiating the above,

$$0 = \delta_{ij} + \frac{\partial \dot{\phi}_i}{\partial \phi_j}(\rho_1 - \rho) + ... \tag{49}$$

This implies that,

$$\frac{\partial \dot{\phi}_i}{\partial \phi_j} \sim \frac{\delta_{ij}}{\rho - \rho_1} + \mathcal{O}(1). \tag{50}$$

This singularity at the initial point is not surprising and is just a reflection of the near "straight -line" behaviour for points very close to the initial point. The problems with the unboundedness of $\frac{\partial \dot{\phi}_i}{\partial \phi_j}$ arise if it has a singularity at any other point than the initial one. This is discussed in the next section.

Which boundary condition one adopts, depends on the problem under consideration. In holographic theories, as discussed earlier, the correspondence between the bulk and boundary theories is given by,

$$< O_i.....O_j > \quad \sim \quad \frac{\delta}{\delta\phi_i}.....\frac{\delta}{\delta\phi_j}S(\phi) \tag{51}$$

where, ϕ_i are the boundary values of the fields and O are the invariant operators in the boundary theory. Thus, the proper choice of boundary conditions in such theories should be such that these derivatives of S are nonvanishing. In general, specifying the conjugate momenta of the fields in the bulk amounts to specifying the nature of the vaccum. For gravitational theories the conjugate momenta relative to $g_{\mu\nu}$ is the energy momentum tensor. We have mentioned earlier that different possible trajectories emnate from the initial point with the slopes $\frac{\delta S}{\delta\phi_i}$ being the distinguishing charactaristic. From a physical point of view, picking a particular value for the slope amounts to choosing a particular vaccum and expectation value of the energy momentum tensor. In fact, in theories including gravitation, it is only the last choice of boundary conditions which ensures even the possibility of a nonvanishing Weyl anomaly. Indeed, it is easy to relate the rate of change of the effective action S (see eq.(4)) to the Weyl anomaly;

$$\frac{dS}{d\rho} \quad = \quad \frac{\partial S}{\partial\rho} + \frac{\delta S}{\delta\phi_i}\dot{\phi}_i. \tag{52}$$

This is just Eq.(13) written in another form. In theories with gravitation, the first term on RHS vanishes and

$$\frac{dS}{d\rho} \quad = \quad \frac{\delta S}{\delta g_{\mu\nu}}\dot{g}_{\mu\nu}. \tag{53}$$

where for simplicity we have only included the gravitational field. Using (18) we may in principle solve for the flow velocity, $\dot{g}_{\mu\nu}$ in terms of the metric and substitute in the above equation. In practice, however this solution gives a non-local expression on the RHS of (53). In some local approximation, the solution of the flow velocity in terms of the metric may be used to obtain the Weyl anomaly at the boundary since then (see example below),

$$RHS(53) \quad \sim \quad \int_d < T^\mu_\mu >, \tag{54}$$

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with $< T^\mu_\mu >$ the trace of the energy momentum tensor. We note that for practical calculations it is easy to evaluate the LHS of (53) in some local approximation. Let us briefly indicate how this works [23] for the conjectured AdS/CFT duality from the bulk point of view.

For the purpose of studying the Weyl anomaly using this method, for the AdS example it will be convenient (though by no means essential) to make a different gauge choice from the one given in (30). We will restore the lapse function N and choose a gauge with,

$$ds^2 = N^2 d\rho^2 + g_{\mu\nu} dx^\mu dx^\nu \tag{55}$$

$$N = \frac{r}{2\rho} \tag{56}$$

In this gauge the AdS boundary is at $\rho = 0$ and the appropriate changes in the equations at the end of the previous section are easy to trace. The most relevant change for us is $\dot{g}_{\mu\nu} \rightarrow N^{-1}\dot{g}_{\mu\nu}$ and this implies that in the case without the scalar fields, the flow and the master equations give in particular,

$$U^2 = \frac{4}{N^2} \frac{d-1}{d} 2\Lambda. \tag{57}$$

Then, in the approximation when the scalar potential U dominates in S,

$$\dot{g}_{\mu\nu} = \frac{2N}{r} g_{\mu\nu} = \frac{1}{\rho} g_{\mu\nu}. \tag{58}$$

Eq.(53) with $\phi_i = g_{\mu\nu}$ in the above mentioned approximation can now can be written as,

$$\rho \frac{dS}{d\rho} = g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} = \int_d < T^\mu_\mu >, \tag{59}$$

It is east to check from the results of ref.[19] that the LHS of the above equation, evaluated at the boundary ($\rho \rightarrow 0$), correctly reproduces the Weyl anomaly there. Note that in general some local invariant counterterms must be added to remove infinities in the action.

It is interesting that a classical calculation provides us with an answer which from the point of view of the boundary theory is purely quantum mechanical. We may attempt to

understand this drawing from Dirac's connection between the classical Poisson brackets (PB) and the quantum mechanical commutator, i.e.,

$$[u, v] = i\frac{h}{2\pi}\{u, v\}_{PB} \qquad (60)$$

Consider Eq.(52) for the gravity case ($\frac{\delta S}{\delta \rho} = 0$), and substitute for $\dot{\phi}_i = \frac{\delta H}{\delta \Pi^i}$. Then noting that S only depends on (ρ, ϕ_i) this equation becomes :

$$\frac{dS}{d\rho} = \{S, H\}_{PB}. \qquad (61)$$

Thus, what we identify with the Weyl anomaly at the boundary in the local limit is :

$$\lim_{\rho \to 0} \rho \{S, H\}_{PB}. \qquad (62)$$

The strong Poisson bracket and commutator analogy gives us the required result. The question of quantum corrections and the connection with the stochastic quantization method will be discussed in detail in a separate publication [14].

4   Limits of Applicability.

In the previous sections we have obtained the equations that govern the effective action and determine its functional form. In deriving the master equation, (13), we assumed the existence and the boundedness of the second partial derivatives of S; in fact, in the derivation we neglected the contributions of the second order derivatives compared to the first. In this section we will study the conditions when this is true. The unboundedness of the second partial derivatives of S imply that the "classical" treatment of this paper is not a good enough approximation. The complete bulk quantum mechanical corrections involving the second derivatives of S would now have to be added on to (13). How this can be implemented is the subject of a separate publication.

It is clear that when the initial point is in the neighbourhood of the terminal point , the master equation (13), is always valid. As the ρ interval increases, it is possible that the

solution to (13) may undergo a qualitative change. It is at such a point that the second derivative of S can become unbounded. In order to quantitatively address this problem, we first relate the unboundedness of the second partial derivatives of S to that of the derivative of the flow velocities and then obtain a differential equation for the latter which does not involve the effective action S but only the "tree" level quantity F and its partial derivatives. Again, throughout the following discussion of boundedness we will replace functional derivatives with ordinary derivatives using Eq.(48) as required.

Indeed, partial differentiation of Eq.(15) w.r.t. ϕ_i gives:

$$F_{,\phi_i\dot\phi_j} \;+\; F_{,\dot\phi_k\dot\phi_j}\frac{\partial\dot\phi_k}{\partial\phi_i} \;\;=\;\; S_{,\phi_i\phi_j}. \tag{63}$$

Thus, $S_{,\phi_i\phi_j}$ will be unbounded if (a) $F_{,\phi_i\dot\phi_j}$ is unbounded, (b) $F_{,\dot\phi_k\dot\phi_j}$ is non-zero and $\frac{\partial\dot\phi_k}{\partial\phi_i}$ is unbounded. Case (a) is trivial and we will focus on case (b) assuming that the second partial derivatives of F are bounded. Then at those points where $\frac{\partial\dot\phi_k}{\partial\phi_i}$ becomes infinite our procedure breaks down and Eqns.(13, 14, 15) are no longer valid. Higher order (quantum) corrections must be included in the form of the second partial derivatives of S. We will now attempt to understand this breakdown more precisely and to discuss the conditions for it to occur.

Having exchanged the problem of unboundedness of the second partial derivative of S for that of the derivative of the flow velocity, we would like to remove all reference to the action S and its derivatives in favor of the known F and its partial derivatives. To this end consider the partial derivative of Eq.(14) w.r.t. ϕ_i for the critical trajectory,

$$F_{,\phi_i} \;\;=\;\; S_{,\rho\phi_i} + S_{,\phi_i\phi_j}\dot\phi_j, \tag{64}$$

where we have used Eq.(15) once to cancel terms involving the derivative of the flow velocities. Next from Eq.(63) we get by taking its total derivative w.r.t. ρ,

$$\left(\frac{d}{d\rho}\left(F_{,\phi_i\dot\phi_j} + F_{,\dot\phi_k\dot\phi_j}\frac{\partial\dot\phi_k}{\partial\phi_i}\right)\right)_{\dot\phi_l} \;\;=\;\; S_{,\rho\phi_i\phi_j} + S_{,\phi_i\phi_j\phi_k}\dot\phi_k. \tag{65}$$

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From Eq.(64) we have :

$$F_{,\phi_i\phi_j} + F_{,\phi_i\dot{\phi}_k}\frac{\partial\dot{\phi}_k}{\partial\phi_j} = S_{,\rho\phi_i\phi_j} + S_{,\phi_i\phi_j\phi_k}\dot{\phi}_k + S_{,\phi_i\phi_k}\frac{\partial\dot{\phi}_k}{\partial\phi_j} \tag{66}$$

Eqns.(64, 65, 66) now give:

$$\left(\frac{d}{d\rho}\left(F_{,\phi_i\dot{\phi}_j} + F_{,\dot{\phi}_k\dot{\phi}_j}\frac{\partial\dot{\phi}_k}{\partial\phi_i}\right)\right)_{\dot{\phi}_l} - F_{,\phi_i\phi_j} + F_{,\dot{\phi}_i\dot{\phi}_l}\frac{\partial\dot{\phi}_l}{\partial\phi_k}\frac{\partial\dot{\phi}_k}{\partial\phi_j} = 0. \tag{67}$$

Eq.(67) is the principal result of this section.

To proceed furthur with an analysis of these equations, we need to specify the boundary conditions for the problem. Suppose that the problem under consideration requires the use of boundary conditions such that the critical curves start at a specified point in (ρ, ϕ_i) space, i.e., at $\rho = \rho_1$, $\phi_i = \phi_{i1}$. Then as discussed in the previous section, near the initial point $\rho \to \rho_1$,

$$\frac{\partial\dot{\phi}_i}{\partial\phi_j} = -\frac{\delta_{ij}}{\rho_1 - \rho} + \mathcal{O}(1). \tag{68}$$

Thus $\frac{\partial\dot{\phi}_i}{\partial\phi_j}$ is singular at the initial point. We have seen that Eq.(67) describes the behaviour of the derivative of the flow velocity at each point on the critical curve. If according to this equation, $\frac{\partial\dot{\phi}_i}{\partial\phi_j}$ becomes singular at any point other than the initial one, then $S_{,\phi_i\phi_j}$ becomes unbounded here.

Consider next the case of the boundary condition such that at $\rho = \rho_1$ and $\phi_i(\rho_1)$ is unspecified. Then we have seen that at the line $\rho = \rho_1$, S and its partial derivatives are zero. Then for this choice of boundary condition, the initial conditions on (67) are such that at $\rho = \rho_1$:

$$\frac{\partial\dot{\phi}_k}{\partial\phi_j}F_{,\dot{\phi}_i\dot{\phi}_k} + F_{,\phi_i\dot{\phi}_j} = 0. \tag{69}$$

Thus we now have a criterion to determine the applicability of the fundamental equation (13).

The above considerations are next applied to the example of a single scalar field. For a single scalar field, the matter lagrangian part of the AdS_{d+1}/CFT_d example discussed in the

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previous section becomes :

$$\mathcal{L}_M = \sqrt{g}\left(V(\phi) + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right),\tag{70}$$

and for this case Eq.(67) for the scalar may be written as :

$$\left(\frac{d}{d\rho}\left(F_{,\phi\dot\phi} + F_{,\dot\phi\dot\phi}\frac{\partial\dot\phi}{\partial\phi}\right)\right)_{\dot\phi} - F_{,\phi\phi} + F_{,\dot\phi\dot\phi}\left(\frac{\partial\dot\phi}{\partial\phi}\right)^2 = 0.\tag{71}$$

In this equation we make a change of variables :

$$\frac{\partial\dot\phi}{\partial\phi} = \frac{\dot W}{W},\tag{72}$$

so that it now becomes :

$$F_{,\dot\phi\dot\phi}\ddot W + \frac{d}{d\rho}F_{,\dot\phi\dot\phi}\dot W + \left(\frac{d}{d\rho}F_{,\phi\dot\phi} - F_{,\phi\phi}\right)W = 0.\tag{73}$$

It is easy to see that the question of the unboundedness of the LHS of Eq.(72) now translates to the existense of zeros of W. Thus with the choice of boundary conditions such that at $\rho = \rho_1$, $\phi = \phi_1$, we know that W has a zero at the initial point and to check if the second derivatives of S are unbounded, we must study the conditions under which Eq.(73) implies other zeros of W. In fact, using Eq.(70), Eq.(73) may be written as ,

$$\ddot W + K(\rho)\dot W - u(\phi)W = 0,\tag{74}$$

where, K is the extrinsic curvature (we are using the gauge choice of Eq.(30)), and $u(\phi) = V_{,\phi\phi}$. It is clear from Eq.(74) that if $u(\phi) \geq 0$, W has at most one zero. More generally, let us make a furthur change of variables from W to Y :

$$W = e^{\int \lambda d\rho}Y.\tag{75}$$

Then the zeros of W are unchanged for finite λ, and Eq.(74) becomes:

$$\ddot Y + (K + 2\lambda)\dot Y + (-u + \lambda^2 + \dot\lambda + \lambda K)Y = 0.\tag{76}$$

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We now choose $-2\lambda = K$ and the above becomes :

$$\ddot{Y} - (u + \frac{K^2}{4} + \frac{1}{2}\dot{K})Y \ = \ 0. \tag{77}$$

Thus, quite generally, we see that if u is negative, but K is large enough and $\dot{K}$ is not too large (and negative), such that,

$$(u + \frac{K^2}{4} + \frac{1}{2}\dot{K}) \geq 0, \tag{78}$$

then, Y and hence W will have only one zero. If not, then the second derivatives of S become unbounded and Eq.(13) is no longer applicable - the quantum mechanical boundary theory cannot just be described by a classical theory in the bulk (tree level supergravity in the case of AdS) . More specifically, in order for the classical description of the bulk to be equivalent to the (quantum) boundary theory, at each point along the critical trajectory the trace of the extrinsic curvature of the foliation at that point must be related to the parameters of the classical theory through the condition (78). One way to satisfy the condition for the AdS case is that the critical trajectory must also minimize the potential V as well. It appears difficult to find another possibility for the AdS case since in this case $K \sim 1/r$ and for holography r is large [4]. Note that in Eqs.(77) and (78) we have retained a term $\dot{K}$ even though it vanishes for the specific case of constant curvature.We do so because in more general situations such a contribution will be present for either sign of u. This is a particularly interesting term and becomes important when K is changing very rapidly at the boundary. Eq.(78) then tells us that when this occurs a holographic description becomes questionable. This is an important conclusion of our analysis. Furthur study of this "non-equilibrium" type situation is, however, beyond the scope of this paper.

5   Domain Walls and Radial Flow

Until now we have discussed the evolution in ρ in a $d+1$ dimensional space-time with no hypersurfaces embedded in it. Consider next a Domain wall in this $d+1$ dimensional space-

time. It has $d - 1$ spatial dimensions (a $d - 1$ brane) and partitions the space-time into different domains. The presence of such domain walls can change the physical properties of the theory, in particular, its spectrum. For example, if in the original theory, the range of ρ was infinite, the domain wall will change this, perhaps making it semi-infinite. Additional normalizeable fluctuations of certain fields (for example the metric in theories with gravity) will therefore make thier appearance where previously (in the absence of the wall) they were absent. In the following we will limit our considerations to a single such domain wall and generalizations can be straightforwardly accomodated.

At domain wall junctions, the flow velocities can become discontinuous. For example, in theories with gravity it is common in such situations that the metric $g_{\mu\nu} \in C^0$ [15]. We are interested in finding out how the partial derivatives of S behave as we cross the domain wall junction. Consider:

$$\frac{d}{d\rho}\left(\frac{\delta S}{\delta \phi_i}\right) = \frac{\partial}{\partial \rho}\frac{\delta S}{\delta \phi_i} + \frac{\delta^2 S}{\delta \phi_i \delta \phi_k}\dot{\phi}_k \tag{79}$$

Comparing this with Eq.(23) we get:

$$\frac{d}{d\rho}\left(\frac{\delta S}{\delta \phi_i}\right) = F_{,\phi_i}. \tag{80}$$

If we denote by Q^+ and Q^- the values of any quantity on the left and right respectively of the domain wall, then :

$$S^+_{,\phi_i} - S^-_{,\phi_i} = \int_{0^-}^{0^+} d\rho F_{,\phi_i}. \tag{81}$$

SImilarly, we have for the partial derivative w.r.t. ρ :

$$\frac{d}{d\rho}\left(\frac{\partial S}{\partial \rho}\right) = \frac{\partial^2 S}{\partial \rho^2} + \frac{\partial}{\partial \rho}\left(\frac{\delta S}{\delta \phi_k}\right)\dot{\phi}_k \tag{82}$$

Then from Eq.(14) we have :

$$F_{,\rho} + F_{,\dot{\phi}_k}\frac{\partial \dot{\phi}_k}{\partial \rho} = \frac{\partial^2 S}{\partial \rho^2} + \frac{\partial}{\partial \rho}\left(\frac{\delta S}{\delta \phi_k}\right)\dot{\phi}_k + S_{,\phi_k}\frac{\partial \dot{\phi}_k}{\partial \rho} \tag{83}$$

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Combining this with Eq.(82) we get:

$$\frac{d}{d\rho}\left(\frac{\delta S}{\delta \rho}\right) = F_{,\rho}: \tag{84}$$

$$S_{,\rho}^{+} - S_{,\rho}^{-} = \int_{0^-}^{0^+} d\rho F_{,\rho}. \tag{85}$$

In deriving the above Eqs.(81, 85) we have implicity assumed the conditions under which the flow velocities may be discontinuous across the domain wall. A condition, obtained from an examination of Eq.(15) is: the change in $F_{,\dot{\phi}_i}$ across the wall is finite (i.e., $F_{,\dot{\phi}_i} \in C$), which can be used together with Eq.(14) to obtain others. Eqs.(14, 17 and 15) and Eq.(81) can be used to find the equations of motion of the various fields at the domain wall. The functional form of S is determined from the first of these and the second gives the equation of motion at the domain wall. We will next apply the above ideas to the case of a domain wall in AdS_{d+1}, and the related Randall-Sundrum type scenario.

The Randall-Sundrum model may be described by the action (with Euclidean signature) :

$$-\int_{d+1} \sqrt{g^{d+1}}(R^{d+1} + 2\Lambda) - 2\int_{d} \sqrt{g}K + T\int_{d} \sqrt{g}, \tag{86}$$

where matter contributions may be added-we are neglecting these for simplicity; and T denotes the brane tension. Randall and Sundrum [13] were able to show (actually for $d = 4$) that for a certain value of T, gravity may be "trapped" on the $d - 1$ brane with an effective Newton constant of $\frac{d-2}{2r}$. Using the picture of radial flow for a bulk AdS theory developed in previous sections we will see how this may be understood simply in terms of a correspondence with a CFT at some finite value of ρ (i.e., away from the boundary of AdS) where the domain wall is situated. Indeed, consider the AdS bulk action of Eq.(29) without the scalars, and to it let us add a brane action of the form,

$$I_{DW} = T\int_{d} d^d x \sqrt{g} = T\int_{d+1} d^d x d\rho \sqrt{g}\delta(\rho); \tag{87}$$

where as discussed before, the domain wall is located at $\rho = 0$. We will next use the junction conditions derived in Eq.(81). Since only the domain wall action contributes to the RHS,

for the present case this equation reduces to

$$S^+_{,g_{\mu\nu}} - S^-_{,g_{\mu\nu}} = T\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\right). \tag{88}$$

In the previous sections we have obtained the form of (the local part of) the effective action S at an arbitrary foliation of AdS with the result:

$$S_{loc} = \int_d d^d x \sqrt{g}(U + \Phi_1 R + \Phi_2 R^2 + \Phi_3 R_{\mu\nu}R^{\mu\nu} + \Phi_4 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + ..., \tag{89}$$

with the following values for the various parameters :

$$U^2 = \frac{4(d-1)}{d}2\Lambda, \tag{90}$$

$$\Phi_1 = \frac{2(d-1)}{d-2}\frac{1}{U}, \tag{91}$$

$$\Phi_2 = -\frac{1}{2}\frac{d}{d-4}\frac{\Phi_1^2}{U}, \tag{92}$$

$$\Phi_3 = 2\frac{d-1}{d-4}\frac{\Phi_1^2}{U}, \tag{93}$$

$$\Phi_4 = 0. \tag{94}$$

Returning to Eq.(88), we now make the simplifying assumption that the entire AdS space has a Z_2 symmetry so that $S^+_{,g_{\mu\nu}} = -S^-_{,g_{\mu\nu}} = S_{,g_{\mu\nu}}$. Then,

$$2S_{,g_{\mu\nu}} = T\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\right). \tag{95}$$

From this it is easy to obtain a variety of conditions; first, we see that in order for the linear terms to cancel (fine tuning of the d dimensional cosmological constant) we must have

$$T = 2U = -4\frac{d-1}{r}, \tag{96}$$

which is just the Randall-Sundrum condition for the brane tension. Secondly, we see that induced gravity on the brane has various contributions , with the leading and next to leading corrections calculated above. From the coefficient Φ_1 we notice the Randall-Sundrum relation for the d dimensional gravitational constant, i.e.,

$$G_d = \frac{d-2}{2r}. \tag{97}$$

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The coefficients $\Phi_i, i = 2, 3, 4$, give the corrections to Newton's law. These give the same results as before (see ref.[13] and the last reference in [16]). From the above we see that in general the induced gravity on the brane is the same as for any foliation of AdS at some value of ρ coincident with the brane position. The generating function for the conjectured dual conformal field theory is related to the non-local part of this effective action S. This conformal field theory on the brane, however, will in general, be different than the one at the boundary of AdS.

6   Conclusions and Outlook

We have developed a variational method for studying the evolution of bulk fields with the radial coordinate in holographic theories. The method appears quite general and is coordinate independant. It is worth pointing out that the entire formalism may be used to derive and to study the exact renormalization group equations in d-dimensions with a fictitious $d+1$th coordinate playing the role of the renormalization group parameter. The resulting equations will be the same as in section 2 which in turn differ from the Polchinski [24] renormalization group equations in that they do not contain terms with the second partial derivatives of the effective action. The conditions when this is a good approximation have been studied in section 4 where certain differential equations were derived. If the solutions of these equations become unbounded then the terms with the second partial derivatives of S must be included. The method of including these will be discussed in a subsequent publication and is related to the stochastic quantization method.

The variational approach has allowed us to look at the boundary conditions for the flow equations in a novel way and to formulate the criteria for thier validity in a quantitative manner. It would be very instructive to determine the conditions for unbounded solutions to Eq.(67) for the AdS case with many scalar fields in a similar manner as for a single scalar. Bulk space-times other than of the AdS type also need to be investigated in detail.

Again the developments in this paper should be useful for this. We have also studied more general situations which include domain walls in bulk space-time. Methods were developed by combining the flow equations with the generalized junction conditions to study aspects of gravity in these brane-world scenarios. Again we are trying to develop other more realistic scenarios than discussed in the paper.

Another advantageous aspect of the variational approach which we have not utilized in this paper involves a systematic study of symmetry properties. Indeed, it is straightforward to obtain Ward-like identities for the effective action. It would be interesting to see what information about the boundary field theory one may obtain when these are combined with the flow equations. Work in these directions is in progress.

7   Acknowledgements

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[17] In the following the index i is a compact notation for Lorentz and internal indices as well as the type of field. To avoid a proliferation of symbols, the sum over i will, in context, also include integration.

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